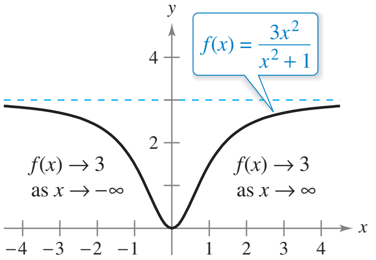
3.5 Limits at Infinity

- **Determine (finite) limits at infinity.**
- **Determine the horizontal asymptotes, if any, of the graph of a function.**
- **Determine infinite limits at infinity.**

Limits at Infinity



The limit of $f(x)$ as x approaches $-\infty$ or ∞ is 3.

Figure 3.32

This section discusses the "end behavior" of a function on an *infinite* interval. Consider the graph of

$$f(x) = \frac{3x^2}{x^2 + 1}$$

as shown in Figure 3.32. Graphically, you can see that the values of $f(x)$ appear to approach 3 as x increases without bound or decreases without bound. You can come to the same conclusions numerically, as shown in the table.

x decreases without bound.						x increases without bound.			

x	$-\infty \leftarrow$	-100	-10	-1	0	1	10	100	$\to \infty$
$f(x)$	$3 \leftarrow$	2.9997	2.9703	1.5	0	1.5	2.9703	2.9997	$\to 3$

$f(x)$ approaches 3. $f(x)$ approaches 3.

The table suggests that the value of $f(x)$ approaches 3 as x increases without bound ($x \to \infty$). Similarly, $f(x)$ approaches 3 as x decreases without bound ($x \to -\infty$). These **limits at infinity** are denoted by

$$\lim_{x \to -\infty} f(x) = 3 \qquad \text{Limit at negative infinity}$$

and

$$\lim_{x \to \infty} f(x) = 3. \qquad \text{Limit at positive infinity}$$

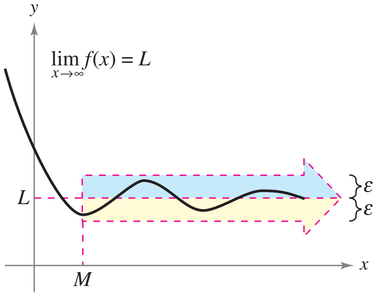
REMARK The statement $\lim_{x \to -\infty} f(x) = L$ or $\lim_{x \to \infty} f(x) = L$ means that the limit exists *and* the limit is equal to L.

To say that a statement is true as x increases *without bound* means that for some (large) real number M, the statement is true for *all* x in the interval $\{x: x > M\}$. The next definition uses this concept.

Definition of Limits at Infinity

Let L be a real number.

1. The statement $\lim_{x \to \infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $M > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > M$.
2. The statement $\lim_{x \to -\infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $N < 0$ such that $|f(x) - L| < \varepsilon$ whenever $x < N$.



$f(x)$ is within ε units of L as $x \to \infty$.

Figure 3.33

The definition of a limit at infinity is shown in Figure 3.33. In this figure, note that for a given positive number ε, there exists a positive number M such that, for $x > M$, the graph of f will lie between the horizontal lines

$$y = L + \varepsilon \quad \text{and} \quad y = L - \varepsilon.$$

Exploration

Use a graphing utility to graph

$$f(x) = \frac{2x^2 + 4x - 6}{3x^2 + 2x - 16}.$$

Describe all the important features of the graph. Can you find a single viewing window that shows all of these features clearly? Explain your reasoning.

What are the horizontal asymptotes of the graph? How far to the right do you have to move on the graph so that the graph is within 0.001 unit of its horizontal asymptote? Explain your reasoning.

Horizontal Asymptotes

In Figure 3.33, the graph of f approaches the line $y = L$ as x increases without bound. The line $y = L$ is called a **horizontal asymptote** of the graph of f.

Definition of a Horizontal Asymptote

The line $y = L$ is a **horizontal asymptote** of the graph of f when

$$\lim_{x\to-\infty} f(x) = L \quad \text{or} \quad \lim_{x\to\infty} f(x) = L.$$

Note that from this definition, it follows that the graph of a *function* of x can have at most two horizontal asymptotes—one to the right and one to the left.

Limits at infinity have many of the same properties of limits discussed in Section 1.3. For example, if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ both exist, then

$$\lim_{x\to\infty} [f(x) + g(x)] = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$$

and

$$\lim_{x\to\infty} [f(x)g(x)] = \Big[\lim_{x\to\infty} f(x)\Big]\Big[\lim_{x\to\infty} g(x)\Big].$$

Similar properties hold for limits at $-\infty$.

When evaluating limits at infinity, the next theorem is helpful.

THEOREM 3.10 Limits at Infinity

If r is a positive rational number and c is any real number, then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0.$$

Furthermore, if x^r is defined when $x < 0$, then

$$\lim_{x\to-\infty} \frac{c}{x^r} = 0.$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 1 Finding a Limit at Infinity

Find the limit: $\displaystyle\lim_{x\to\infty} \left(5 - \frac{2}{x^2}\right)$.

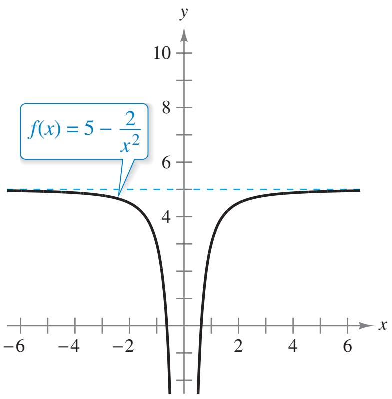
Solution Using Theorem 3.10, you can write

$$\begin{aligned}
\lim_{x\to\infty} \left(5 - \frac{2}{x^2}\right) &= \lim_{x\to\infty} 5 - \lim_{x\to\infty} \frac{2}{x^2} && \text{Property of limits} \\
&= 5 - 0 \\
&= 5.
\end{aligned}$$

So, the line $y = 5$ is a horizontal asymptote to the right. By finding the limit

$$\lim_{x\to-\infty} \left(5 - \frac{2}{x^2}\right) \qquad \text{Limit as } x \to -\infty.$$

you can see that $y = 5$ is also a horizontal asymptote to the left. The graph of the function is shown in Figure 3.34.



$y = 5$ is a horizontal asymptote.
Figure 3.34

EXAMPLE 2 Finding a Limit at Infinity

Find the limit: $\lim_{x\to\infty} \frac{2x-1}{x+1}$.

Solution Note that both the numerator and the denominator approach infinity as x approaches infinity.

$$\lim_{x\to\infty} \frac{2x-1}{x+1} \quad \begin{cases} \lim_{x\to\infty} (2x-1) \to \infty \\ \lim_{x\to\infty} (x+1) \to \infty \end{cases}$$

This results in $\frac{\infty}{\infty}$, an **indeterminate form.** To resolve this problem, you can divide both the numerator and the denominator by x. After dividing, the limit may be evaluated as shown.

$$\lim_{x\to\infty} \frac{2x-1}{x+1} = \lim_{x\to\infty} \frac{\frac{2x-1}{x}}{\frac{x+1}{x}} \qquad \text{Divide numerator and denominator by } x.$$

$$= \lim_{x\to\infty} \frac{2-\frac{1}{x}}{1+\frac{1}{x}} \qquad \text{Simplify.}$$

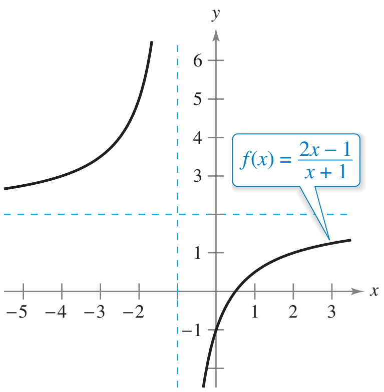
$$= \frac{\lim_{x\to\infty} 2 - \lim_{x\to\infty} \frac{1}{x}}{\lim_{x\to\infty} 1 + \lim_{x\to\infty} \frac{1}{x}} \qquad \text{Take limits of numerator and denominator.}$$

$$= \frac{2-0}{1+0} \qquad \text{Apply Theorem 3.10.}$$

$$= 2$$

So, the line $y = 2$ is a horizontal asymptote to the right. By taking the limit as $x \to -\infty$, you can see that $y = 2$ is also a horizontal asymptote to the left. The graph of the function is shown in Figure 3.35.

REMARK When you encounter an indeterminate form such as the one in Example 2, you should divide the numerator and denominator by the highest power of x in the *denominator*.



$y = 2$ is a horizontal asymptote.
Figure 3.35

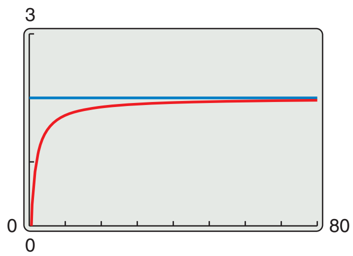
TECHNOLOGY You can test the reasonableness of the limit found in Example 2 by evaluating $f(x)$ for a few large positive values of x. For instance,

$$f(100) \approx 1.9703, \quad f(1000) \approx 1.9970, \quad \text{and} \quad f(10{,}000) \approx 1.9997.$$

Another way to test the reasonableness of the limit is to use a graphing utility. For instance, in Figure 3.36, the graph of

$$f(x) = \frac{2x-1}{x+1}$$

is shown with the horizontal line $y = 2$. Note that as x increases, the graph of f moves closer and closer to its horizontal asymptote.



As x increases, the graph of f moves closer and closer to the line $y = 2$.
Figure 3.36

EXAMPLE 3 A Comparison of Three Rational Functions

▷ *See LarsonCalculus.com for an interactive version of this type of example.*

Find each limit.

a. $\displaystyle\lim_{x\to\infty}\frac{2x+5}{3x^2+1}$ **b.** $\displaystyle\lim_{x\to\infty}\frac{2x^2+5}{3x^2+1}$ **c.** $\displaystyle\lim_{x\to\infty}\frac{2x^3+5}{3x^2+1}$

Solution In each case, attempting to evaluate the limit produces the indeterminate form ∞/∞.

a. Divide both the numerator and the denominator by x^2.

$$\lim_{x\to\infty}\frac{2x+5}{3x^2+1}=\lim_{x\to\infty}\frac{(2/x)+(5/x^2)}{3+(1/x^2)}=\frac{0+0}{3+0}=\frac{0}{3}=0$$

b. Divide both the numerator and the denominator by x^2.

$$\lim_{x\to\infty}\frac{2x^2+5}{3x^2+1}=\lim_{x\to\infty}\frac{2+(5/x^2)}{3+(1/x^2)}=\frac{2+0}{3+0}=\frac{2}{3}$$

c. Divide both the numerator and the denominator by x^2.

$$\lim_{x\to\infty}\frac{2x^3+5}{3x^2+1}=\lim_{x\to\infty}\frac{2x+(5/x^2)}{3+(1/x^2)}=\frac{\infty}{3}$$

You can conclude that the limit *does not exist* because the numerator increases without bound while the denominator approaches 3.

MARIA GAETANA AGNESI (1718–1799)

Agnesi was one of a handful of women to receive credit for significant contributions to mathematics before the twentieth century. In her early twenties, she wrote the first text that included both differential and integral calculus. By age 30, she was an honorary member of the faculty at the University of Bologna.

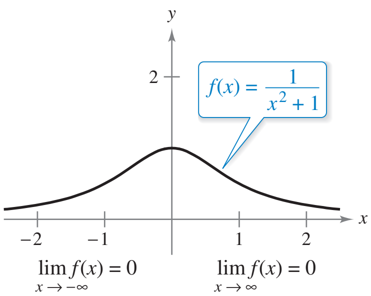
See LarsonCalculus.com to read more of this biography.

For more information on the contributions of women to mathematics, see the article "Why Women Succeed in Mathematics" by Mona Fabricant, Sylvia Svitak, and Patricia Clark Kenschaft in *Mathematics Teacher*. To view this article, go to *MathArticles.com*.

Example 3 suggests the guidelines below for finding limits at infinity of rational functions. Use these guidelines to check the results in Example 3.

GUIDELINES FOR FINDING LIMITS AT $\pm\infty$ OF RATIONAL FUNCTIONS

1. If the degree of the numerator is *less than* the degree of the denominator, then the limit of the rational function is 0.
2. If the degree of the numerator is *equal to* the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.
3. If the degree of the numerator is *greater than* the degree of the denominator, then the limit of the rational function does not exist.



f has a horizontal asymptote at $y=0$.
Figure 3.37

The guidelines for finding limits at infinity of rational functions seem reasonable when you consider that for large values of x, the highest-power term of the rational function is the most "influential" in determining the limit. For instance,

$$\lim_{x\to\infty}\frac{1}{x^2+1}$$

is 0 because the denominator overpowers the numerator as x increases or decreases without bound, as shown in Figure 3.37.

The function shown in Figure 3.37 is a special case of a type of curve studied by the Italian mathematician Maria Gaetana Agnesi. The general form of this function is

$$f(x)=\frac{8a^3}{x^2+4a^2}\qquad\text{Witch of Agnesi}$$

and, through a mistranslation of the Italian word *vertéré*, the curve has come to be known as the Witch of Agnesi. Agnesi's work with this curve first appeared in a comprehensive text on calculus that was published in 1748.

The Granger Collection, New York

In Figure 3.37, you can see that the function

$$f(x) = \frac{1}{x^2 + 1}$$

approaches the same horizontal asymptote to the right and to the left. This is always true of rational functions. Functions that are not rational, however, may approach different horizontal asymptotes to the right and to the left. This is demonstrated in Example 4.

EXAMPLE 4 A Function with Two Horizontal Asymptotes

Find each limit.

a. $\displaystyle\lim_{x\to\infty} \frac{3x - 2}{\sqrt{2x^2 + 1}}$ **b.** $\displaystyle\lim_{x\to-\infty} \frac{3x - 2}{\sqrt{2x^2 + 1}}$

Solution

a. For $x > 0$, you can write $x = \sqrt{x^2}$. So, dividing both the numerator and the denominator by x produces

$$\frac{3x - 2}{\sqrt{2x^2 + 1}} = \frac{\dfrac{3x - 2}{x}}{\dfrac{\sqrt{2x^2 + 1}}{\sqrt{x^2}}} = \frac{3 - \dfrac{2}{x}}{\sqrt{\dfrac{2x^2 + 1}{x^2}}} = \frac{3 - \dfrac{2}{x}}{\sqrt{2 + \dfrac{1}{x^2}}}$$

and you can take the limit as follows.

$$\lim_{x\to\infty} \frac{3x - 2}{\sqrt{2x^2 + 1}} = \lim_{x\to\infty} \frac{3 - \dfrac{2}{x}}{\sqrt{2 + \dfrac{1}{x^2}}} = \frac{3 - 0}{\sqrt{2 + 0}} = \frac{3}{\sqrt{2}}$$

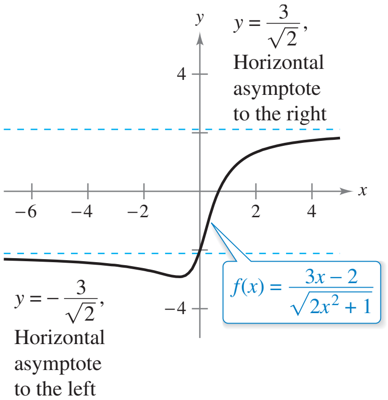
b. For $x < 0$, you can write $x = -\sqrt{x^2}$. So, dividing both the numerator and the denominator by x produces

$$\frac{3x - 2}{\sqrt{2x^2 + 1}} = \frac{\dfrac{3x - 2}{x}}{\dfrac{\sqrt{2x^2 + 1}}{-\sqrt{x^2}}} = \frac{3 - \dfrac{2}{x}}{-\sqrt{\dfrac{2x^2 + 1}{x^2}}} = \frac{3 - \dfrac{2}{x}}{-\sqrt{2 + \dfrac{1}{x^2}}}$$

and you can take the limit as follows.

$$\lim_{x\to-\infty} \frac{3x - 2}{\sqrt{2x^2 + 1}} = \lim_{x\to-\infty} \frac{3 - \dfrac{2}{x}}{-\sqrt{2 + \dfrac{1}{x^2}}} = \frac{3 - 0}{-\sqrt{2 + 0}} = -\frac{3}{\sqrt{2}}$$

The graph of $f(x) = (3x - 2)/\sqrt{2x^2 + 1}$ is shown in Figure 3.38.

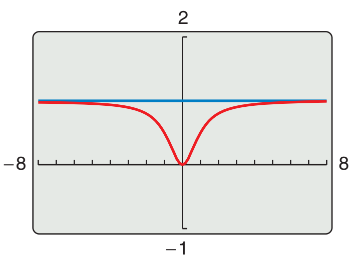


Functions that are not rational may have different right and left horizontal asymptotes.
Figure 3.38

TECHNOLOGY PITFALL If you use a graphing utility to estimate a limit, be sure that you also confirm the estimate analytically—the pictures shown by a graphing utility can be misleading. For instance, Figure 3.39 shows one view of the graph of

$$y = \frac{2x^3 + 1000x^2 + x}{x^3 + 1000x^2 + x + 1000}.$$

From this view, one could be convinced that the graph has $y = 1$ as a horizontal asymptote. An analytical approach shows that the horizontal asymptote is actually $y = 2$. Confirm this by enlarging the viewing window on the graphing utility.



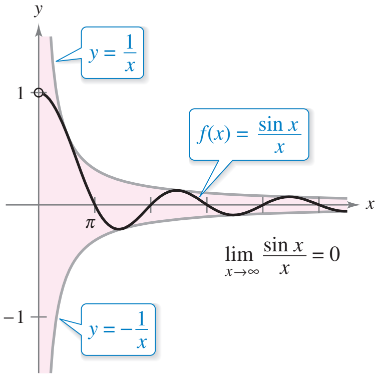
The horizontal asymptote appears to be the line $y = 1$, but it is actually the line $y = 2$.
Figure 3.39

In Section 1.3 (Example 9), you saw how the Squeeze Theorem can be used to evaluate limits involving trigonometric functions. This theorem is also valid for limits at infinity.

EXAMPLE 5 Limits Involving Trigonometric Functions

Find each limit.

a. $\lim_{x\to\infty} \sin x$ **b.** $\lim_{x\to\infty} \frac{\sin x}{x}$



As x increases without bound, $f(x)$ approaches 0.
Figure 3.40

Solution

a. As x approaches infinity, the sine function oscillates between 1 and -1. So, this limit does not exist.

b. Because $-1 \le \sin x \le 1$, it follows that for $x > 0$,

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

where

$$\lim_{x\to\infty}\left(-\frac{1}{x}\right) = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{1}{x} = 0.$$

So, by the Squeeze Theorem, you can obtain

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$

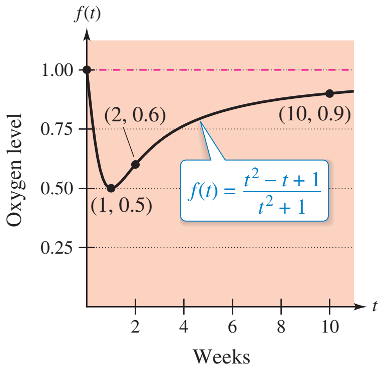
as shown in Figure 3.40.

EXAMPLE 6 Oxygen Level in a Pond

Let $f(t)$ measure the level of oxygen in a pond, where $f(t) = 1$ is the normal (unpolluted) level and the time t is measured in weeks. When $t = 0$, organic waste is dumped into the pond, and as the waste material oxidizes, the level of oxygen in the pond is

$$f(t) = \frac{t^2 - t + 1}{t^2 + 1}.$$

What percent of the normal level of oxygen exists in the pond after 1 week? After 2 weeks? After 10 weeks? What is the limit as t approaches infinity?



The level of oxygen in a pond approaches the normal level of 1 as t approaches ∞.
Figure 3.41

Solution When $t = 1$, 2, and 10, the levels of oxygen are as shown.

$$f(1) = \frac{1^2 - 1 + 1}{1^2 + 1} = \frac{1}{2} = 50\% \qquad \text{1 week}$$

$$f(2) = \frac{2^2 - 2 + 1}{2^2 + 1} = \frac{3}{5} = 60\% \qquad \text{2 weeks}$$

$$f(10) = \frac{10^2 - 10 + 1}{10^2 + 1} = \frac{91}{101} \approx 90.1\% \qquad \text{10 weeks}$$

To find the limit as t approaches infinity, you can use the guidelines on page 198, or you can divide the numerator and the denominator by t^2 to obtain

$$\lim_{t\to\infty} \frac{t^2 - t + 1}{t^2 + 1} = \lim_{t\to\infty} \frac{1 - (1/t) + (1/t^2)}{1 + (1/t^2)} = \frac{1 - 0 + 0}{1 + 0} = 1 = 100\%.$$

See Figure 3.41.

Infinite Limits at Infinity

Many functions do not approach a finite limit as x increases (or decreases) without bound. For instance, no polynomial function has a finite limit at infinity. The next definition is used to describe the behavior of polynomial and other functions at infinity.

REMARK Determining whether a function has an infinite limit at infinity is useful in analyzing the "end behavior" of its graph. You will see examples of this in Section 3.6 on curve sketching.

Definition of Infinite Limits at Infinity

Let f be a function defined on the interval (a, ∞).

1. The statement $\lim_{x\to\infty} f(x) = \infty$ means that for each positive number M, there is a corresponding number $N > 0$ such that $f(x) > M$ whenever $x > N$.
2. The statement $\lim_{x\to\infty} f(x) = -\infty$ means that for each negative number M, there is a corresponding number $N > 0$ such that $f(x) < M$ whenever $x > N$.

Similar definitions can be given for the statements

$$\lim_{x\to-\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = -\infty.$$

EXAMPLE 7 Finding Infinite Limits at Infinity

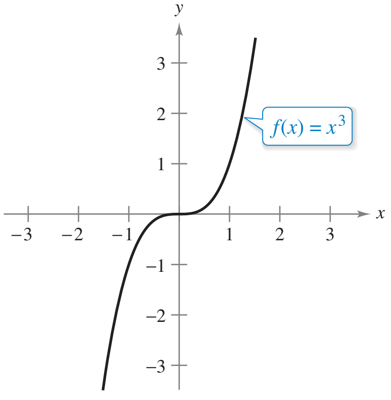


Figure 3.42

Find each limit.

a. $\lim_{x\to\infty} x^3$ **b.** $\lim_{x\to-\infty} x^3$

Solution

a. As x increases without bound, x^3 also increases without bound. So, you can write

$$\lim_{x\to\infty} x^3 = \infty.$$

b. As x decreases without bound, x^3 also decreases without bound. So, you can write

$$\lim_{x\to-\infty} x^3 = -\infty.$$

The graph of $f(x) = x^3$ in Figure 3.42 illustrates these two results. These results agree with the Leading Coefficient Test for polynomial functions as described in Section P.3.

EXAMPLE 8 Finding Infinite Limits at Infinity

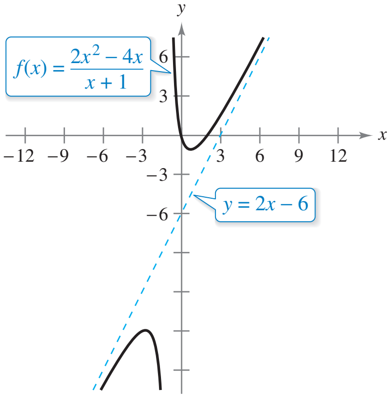


Figure 3.43

Find each limit.

a. $\lim_{x\to\infty} \dfrac{2x^2 - 4x}{x + 1}$ **b.** $\lim_{x\to-\infty} \dfrac{2x^2 - 4x}{x + 1}$

Solution One way to evaluate each of these limits is to use long division to rewrite the improper rational function as the sum of a polynomial and a rational function.

$$\textbf{a.}\ \lim_{x\to\infty} \frac{2x^2 - 4x}{x + 1} = \lim_{x\to\infty}\left(2x - 6 + \frac{6}{x + 1}\right) = \infty$$

$$\textbf{b.}\ \lim_{x\to-\infty} \frac{2x^2 - 4x}{x + 1} = \lim_{x\to-\infty}\left(2x - 6 + \frac{6}{x + 1}\right) = -\infty$$

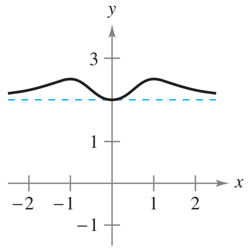
The statements above can be interpreted as saying that as x approaches $\pm\infty$, the function $f(x) = (2x^2 - 4x)/(x + 1)$ behaves like the function $g(x) = 2x - 6$. In Section 3.6, you will see that this is graphically described by saying that the line $y = 2x - 6$ is a slant asymptote of the graph of f, as shown in Figure 3.43.

3.5 Exercises

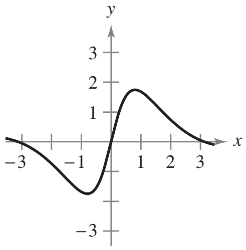
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Matching In Exercises 1–6, match the function with one of the graphs [(a), (b), (c), (d), (e), or (f)] using horizontal asymptotes as an aid.

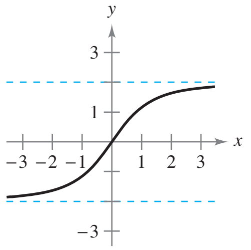
(a)



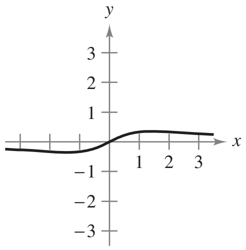
(b)



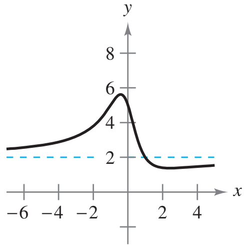
(c)



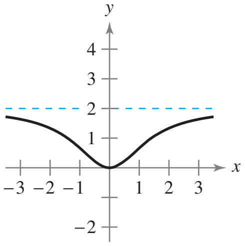
(d)



(e)



(f)



1. $f(x) = \dfrac{2x^2}{x^2+2}$

2. $f(x) = \dfrac{2x}{\sqrt{x^2+2}}$

3. $f(x) = \dfrac{x}{x^2+2}$

4. $f(x) = 2 + \dfrac{x^2}{x^4+1}$

5. $f(x) = \dfrac{4\sin x}{x^2+1}$

6. $f(x) = \dfrac{2x^2-3x+5}{x^2+1}$

Numerical and Graphical Analysis In Exercises 7–12, use a graphing utility to complete the table and estimate the limit as x approaches infinity. Then use a graphing utility to graph the function and estimate the limit graphically.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$							

7. $f(x) = \dfrac{4x+3}{2x-1}$

8. $f(x) = \dfrac{2x^2}{x+1}$

9. $f(x) = \dfrac{-6x}{\sqrt{4x^2+5}}$

10. $f(x) = \dfrac{10}{\sqrt{2x^2-1}}$

11. $f(x) = 5 - \dfrac{1}{x^2+1}$

12. $f(x) = 4 + \dfrac{3}{x^2+2}$

Finding Limits at Infinity In Exercises 13 and 14, find $\lim\limits_{x\to\infty} h(x)$, if possible.

13. $f(x) = 5x^3 - 3x^2 + 10x$

(a) $h(x) = \dfrac{f(x)}{x^2}$

(b) $h(x) = \dfrac{f(x)}{x^3}$

(c) $h(x) = \dfrac{f(x)}{x^4}$

14. $f(x) = -4x^2 + 2x - 5$

(a) $h(x) = \dfrac{f(x)}{x}$

(b) $h(x) = \dfrac{f(x)}{x^2}$

(c) $h(x) = \dfrac{f(x)}{x^3}$

Finding Limits at Infinity In Exercises 15–18, find each limit, if possible.

15. (a) $\lim\limits_{x\to\infty} \dfrac{x^2+2}{x^3-1}$

(b) $\lim\limits_{x\to\infty} \dfrac{x^2+2}{x^2-1}$

(c) $\lim\limits_{x\to\infty} \dfrac{x^2+2}{x-1}$

16. (a) $\lim\limits_{x\to\infty} \dfrac{3-2x}{3x^3-1}$

(b) $\lim\limits_{x\to\infty} \dfrac{3-2x}{3x-1}$

(c) $\lim\limits_{x\to\infty} \dfrac{3-2x^2}{3x-1}$

17. (a) $\lim\limits_{x\to\infty} \dfrac{5-2x^{3/2}}{3x^2-4}$

(b) $\lim\limits_{x\to\infty} \dfrac{5-2x^{3/2}}{3x^{3/2}-4}$

(c) $\lim\limits_{x\to\infty} \dfrac{5-2x^{3/2}}{3x-4}$

18. (a) $\lim\limits_{x\to\infty} \dfrac{5x^{3/2}}{4x^2+1}$

(b) $\lim\limits_{x\to\infty} \dfrac{5x^{3/2}}{4x^{3/2}+1}$

(c) $\lim\limits_{x\to\infty} \dfrac{5x^{3/2}}{4\sqrt{x}+1}$

Finding a Limit In Exercises 19–38, find the limit.

19. $\lim\limits_{x\to\infty} \left(4 + \dfrac{3}{x}\right)$

20. $\lim\limits_{x\to-\infty} \left(\dfrac{5}{x} - \dfrac{x}{3}\right)$

21. $\lim\limits_{x\to\infty} \dfrac{2x-1}{3x+2}$

22. $\lim\limits_{x\to-\infty} \dfrac{4x^2+5}{x^2+3}$

23. $\lim\limits_{x\to\infty} \dfrac{x}{x^2-1}$

24. $\lim\limits_{x\to\infty} \dfrac{5x^3+1}{10x^3-3x^2+7}$

25. $\lim\limits_{x\to-\infty} \dfrac{5x^2}{x+3}$

26. $\lim\limits_{x\to-\infty} \dfrac{x^3-4}{x^2+1}$

27. $\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2-x}}$

28. $\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2+1}}$

29. $\lim\limits_{x\to-\infty} \dfrac{2x+1}{\sqrt{x^2-x}}$

30. $\lim\limits_{x\to\infty} \dfrac{5x^2+2}{\sqrt{x^2+3}}$

31. $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2-1}}{2x-1}$

32. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x^4-1}}{x^3-1}$

33. $\lim\limits_{x\to\infty} \dfrac{x+1}{(x^2+1)^{1/3}}$

34. $\lim\limits_{x\to-\infty} \dfrac{2x}{(x^6-1)^{1/3}}$

35. $\lim\limits_{x\to\infty} \dfrac{1}{2x+\sin x}$

36. $\lim\limits_{x\to\infty} \cos\dfrac{1}{x}$

37. $\lim\limits_{x\to\infty} \dfrac{\sin 2x}{x}$

38. $\lim\limits_{x\to\infty} \dfrac{x-\cos x}{x}$

Horizontal Asymptotes **In Exercises 39–42, use a graphing utility to graph the function and identify any horizontal asymptotes.**

39. $f(x) = \dfrac{|x|}{x+1}$

40. $f(x) = \dfrac{|3x+2|}{x-2}$

41. $f(x) = \dfrac{3x}{\sqrt{x^2+2}}$

42. $f(x) = \dfrac{\sqrt{9x^2-2}}{2x+1}$

Finding a Limit **In Exercises 43 and 44, find the limit. (*Hint:* Let $x = 1/t$ and find the limit as $t \to 0^+$.)**

43. $\lim\limits_{x\to\infty} x \sin \dfrac{1}{x}$

44. $\lim\limits_{x\to\infty} x \tan \dfrac{1}{x}$

Finding a Limit **In Exercises 45–48, find the limit. (*Hint*: Treat the expression as a fraction whose denominator is 1, and rationalize the numerator.) Use a graphing utility to verify your result.**

45. $\lim\limits_{x\to-\infty} \left(x + \sqrt{x^2+3}\right)$

46. $\lim\limits_{x\to\infty} \left(x - \sqrt{x^2+x}\right)$

47. $\lim\limits_{x\to-\infty} \left(3x + \sqrt{9x^2-x}\right)$

48. $\lim\limits_{x\to\infty} \left(4x - \sqrt{16x^2-x}\right)$

Numerical, Graphical, and Analytic Analysis **In Exercises 49–52, use a graphing utility to complete the table and estimate the limit as x approaches infinity. Then use a graphing utility to graph the function and estimate the limit. Finally, find the limit analytically and compare your results with the estimates.**

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$							

49. $f(x) = x - \sqrt{x(x-1)}$

50. $f(x) = x^2 - x\sqrt{x(x-1)}$

51. $f(x) = x \sin \dfrac{1}{2x}$

52. $f(x) = \dfrac{x+1}{x\sqrt{x}}$

WRITING ABOUT CONCEPTS

Writing **In Exercises 53 and 54, describe in your own words what the statement means.**

53. $\lim\limits_{x\to\infty} f(x) = 4$

54. $\lim\limits_{x\to-\infty} f(x) = 2$

55. **Sketching a Graph** Sketch a graph of a differentiable function f that satisfies the following conditions and has $x = 2$ as its only critical number.

$f'(x) < 0$ for $x < 2$

$f'(x) > 0$ for $x > 2$

$\lim\limits_{x\to-\infty} f(x) = 6$

$\lim\limits_{x\to\infty} f(x) = 6$

56. **Points of Inflection** Is it possible to sketch a graph of a function that satisfies the conditions of Exercise 55 and has *no* points of inflection? Explain.

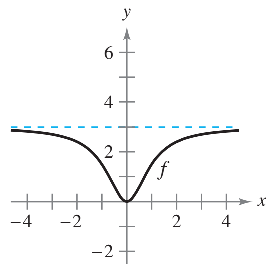
WRITING ABOUT CONCEPTS (continued)

57. **Using Symmetry to Find Limits** If f is a continuous function such that $\lim\limits_{x\to\infty} f(x) = 5$, find, if possible, $\lim\limits_{x\to-\infty} f(x)$ for each specified condition.

(a) The graph of f is symmetric with respect to the y-axis.

(b) The graph of f is symmetric with respect to the origin.

58. **A Function and Its Derivative** The graph of a function f is shown below. To print an enlarged copy of the graph, go to *MathGraphs.com.*



(a) Sketch f'.

(b) Use the graphs to estimate $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to\infty} f'(x)$.

(c) Explain the answers you gave in part (b).

Sketching a Graph **In Exercises 59–74, sketch the graph of the equation using extrema, intercepts, symmetry, and asymptotes. Then use a graphing utility to verify your result.**

59. $y = \dfrac{x}{1-x}$

60. $y = \dfrac{x-4}{x-3}$

61. $y = \dfrac{x+1}{x^2-4}$

62. $y = \dfrac{2x}{9-x^2}$

63. $y = \dfrac{x^2}{x^2+16}$

64. $y = \dfrac{2x^2}{x^2-4}$

65. $xy^2 = 9$

66. $x^2y = 9$

67. $y = \dfrac{3x}{x-1}$

68. $y = \dfrac{3x}{1-x^2}$

69. $y = 2 - \dfrac{3}{x^2}$

70. $y = 1 - \dfrac{1}{x}$

71. $y = 3 + \dfrac{2}{x}$

72. $y = \dfrac{4}{x^2} + 1$

73. $y = \dfrac{x^3}{\sqrt{x^2-4}}$

74. $y = \dfrac{x}{\sqrt{x^2-4}}$

Analyzing a Graph Using Technology **In Exercises 75–82, use a computer algebra system to analyze the graph of the function. Label any extrema and/or asymptotes that exist.**

75. $f(x) = 9 - \dfrac{5}{x^2}$

76. $f(x) = \dfrac{1}{x^2-x-2}$

77. $f(x) = \dfrac{x-2}{x^2-4x+3}$

78. $f(x) = \dfrac{x+1}{x^2+x+1}$

79. $f(x) = \dfrac{3x}{\sqrt{4x^2 + 1}}$

80. $g(x) = \dfrac{2x}{\sqrt{3x^2 + 1}}$

81. $g(x) = \sin\left(\dfrac{x}{x - 2}\right), \quad x > 3$

82. $f(x) = \dfrac{2 \sin 2x}{x}$

Comparing Functions **In Exercises 83 and 84, (a) use a graphing utility to graph f and g in the same viewing window, (b) verify algebraically that f and g represent the same function, and (c) zoom out sufficiently far so that the graph appears as a line. What equation does this line appear to have? (Note that the points at which the function is not continuous are not readily seen when you zoom out.)**

83. $f(x) = \dfrac{x^3 - 3x^2 + 2}{x(x - 3)}$

$g(x) = x + \dfrac{2}{x(x - 3)}$

84. $f(x) = -\dfrac{x^3 - 2x^2 + 2}{2x^2}$

$g(x) = -\dfrac{1}{2}x + 1 - \dfrac{1}{x^2}$

85. Engine Efficiency

The efficiency of an internal combustion engine is

$$\text{Efficiency } (\%) = 100\left[1 - \frac{1}{(v_1/v_2)^c}\right]$$

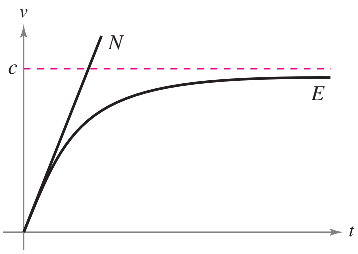
where v_1/v_2 is the ratio of the uncompressed gas to the compressed gas and c is a positive constant dependent on the engine design. Find the limit of the efficiency as the compression ratio approaches infinity.

86. Average Cost A business has a cost of $C = 0.5x + 500$ for producing x units. The average cost per unit is

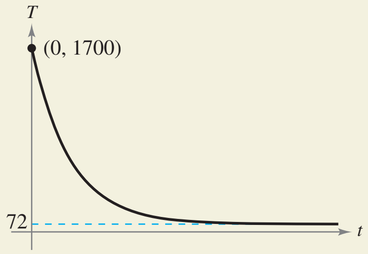
$$\overline{C} = \frac{C}{x}.$$

Find the limit of $\overline{C}$ as x approaches infinity.

87. Physics Newton's First Law of Motion and Einstein's Special Theory of Relativity differ concerning a particle's behavior as its velocity approaches the speed of light c. In the graph, functions N and E represent the velocity v, with respect to time t, of a particle accelerated by a constant force as predicted by Newton and Einstein, respectively. Write limit statements that describe these two theories.



88. HOW DO YOU SEE IT? The graph shows the temperature T, in degrees Fahrenheit, of molten glass t seconds after it is removed from a kiln.



(a) Find $\lim_{t \to 0^+} T$. What does this limit represent?

(b) Find $\lim_{t \to \infty} T$. What does this limit represent?

(c) Will the temperature of the glass ever actually reach room temperature? Why?

89. Modeling Data The average typing speeds S (in words per minute) of a typing student after t weeks of lessons are shown in the table.

t	5	10	15	20	25	30
S	28	56	79	90	93	94

A model for the data is $S = \dfrac{100t^2}{65 + t^2}, \; t > 0.$

(a) Use a graphing utility to plot the data and graph the model.

(b) Does there appear to be a limiting typing speed? Explain.

90. Modeling Data A heat probe is attached to the heat exchanger of a heating system. The temperature T (in degrees Celsius) is recorded t seconds after the furnace is started. The results for the first 2 minutes are recorded in the table.

t	0	15	30	45	60
T	25.2°	36.9°	45.5°	51.4°	56.0°

t	75	90	105	120
T	59.6°	62.0°	64.0°	65.2°

(a) Use the regression capabilities of a graphing utility to find a model of the form $T_1 = at^2 + bt + c$ for the data.

(b) Use a graphing utility to graph T_1.

(c) A rational model for the data is

$$T_2 = \frac{1451 + 86t}{58 + t}.$$

Use a graphing utility to graph T_2.

(d) Find $T_1(0)$ and $T_2(0)$.

(e) Find $\lim_{t \to \infty} T_2$.

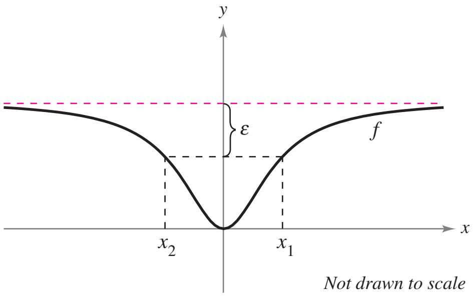
(f) Interpret the result in part (e) in the context of the problem. Is it possible to do this type of analysis using T_1? Explain.

Straight 8 Photography/Shutterstock.com

91. Using the Definition of Limits at Infinity The graph of

$$f(x) = \frac{2x^2}{x^2 + 2}$$

is shown.



Not drawn to scale

(a) Find $L = \lim_{x\to\infty} f(x)$.

(b) Determine x_1 and x_2 in terms of ε.

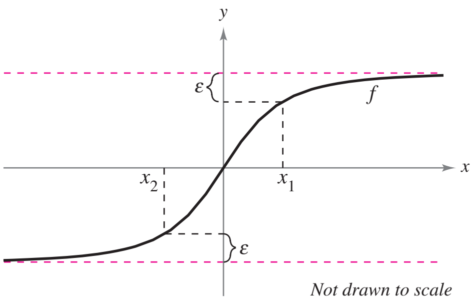
(c) Determine M, where $M > 0$, such that $|f(x) - L| < \varepsilon$ for $x > M$.

(d) Determine N, where $N < 0$, such that $|f(x) - L| < \varepsilon$ for $x < N$.

92. Using the Definition of Limits at Infinity The graph of

$$f(x) = \frac{6x}{\sqrt{x^2 + 2}}$$

is shown.



Not drawn to scale

(a) Find $L = \lim_{x\to\infty} f(x)$ and $K = \lim_{x\to-\infty} f(x)$.

(b) Determine x_1 and x_2 in terms of ε.

(c) Determine M, where $M > 0$, such that $|f(x) - L| < \varepsilon$ for $x > M$.

(d) Determine N, where $N < 0$, such that $|f(x) - K| < \varepsilon$ for $x < N$.

93. Using the Definition of Limits at Infinity Consider

$$\lim_{x\to\infty} \frac{3x}{\sqrt{x^2 + 3}}.$$

(a) Use the definition of limits at infinity to find values of M that correspond to $\varepsilon = 0.5$.

(b) Use the definition of limits at infinity to find values of M that correspond to $\varepsilon = 0.1$.

94. Using the Definition of Limits at Infinity Consider

$$\lim_{x\to-\infty} \frac{3x}{\sqrt{x^2 + 3}}.$$

(a) Use the definition of limits at infinity to find values of N that correspond to $\varepsilon = 0.5$.

(b) Use the definition of limits at infinity to find values of N that correspond to $\varepsilon = 0.1$.

Proof In Exercises 95–98, use the definition of limits at infinity to prove the limit.

95. $\lim_{x\to\infty} \frac{1}{x^2} = 0$

96. $\lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$

97. $\lim_{x\to-\infty} \frac{1}{x^3} = 0$

98. $\lim_{x\to-\infty} \frac{1}{x - 2} = 0$

99. Distance A line with slope m passes through the point $(0, 4)$.

(a) Write the shortest distance d between the line and the point $(3, 1)$ as a function of m.

(b) Use a graphing utility to graph the equation in part (a).

(c) Find $\lim_{m\to\infty} d(m)$ and $\lim_{m\to-\infty} d(m)$. Interpret the results geometrically.

100. Distance A line with slope m passes through the point $(0, -2)$.

(a) Write the shortest distance d between the line and the point $(4, 2)$ as a function of m.

(b) Use a graphing utility to graph the equation in part (a).

(c) Find $\lim_{m\to\infty} d(m)$ and $\lim_{m\to-\infty} d(m)$. Interpret the results geometrically.

101. Proof Prove that if

$$p(x) = a_n x^n + \cdots + a_1 x + a_0$$

and

$$q(x) = b_m x^m + \cdots + b_1 x + b_0$$

where $a_n \neq 0$ and $b_m \neq 0$, then

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \begin{cases} 0, & n < m \\ \dfrac{a_n}{b_m}, & n = m. \\ \pm\infty, & n > m \end{cases}$$

102. Proof Use the definition of infinite limits at infinity to prove that $\lim_{x\to\infty} x^3 = \infty$.

True or False? In Exercises 103 and 104, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

103. If $f'(x) > 0$ for all real numbers x, then f increases without bound.

104. If $f''(x) < 0$ for all real numbers x, then f decreases without bound.